2 pole Butterworth Filter

$\omega_0 = 1575.42\ \text{MHz} \cdot 2\pi$

$w = 6\%$

$Z_0 = 50\Omega$

N	g_1	g_2	g_3
1	2.0000	1.0000	
2	1.4142	1.4142	1.0000

Step 1:

We will start with a shunt capacitor.

$C = 1.4141$ will get transformed to a parallel LC

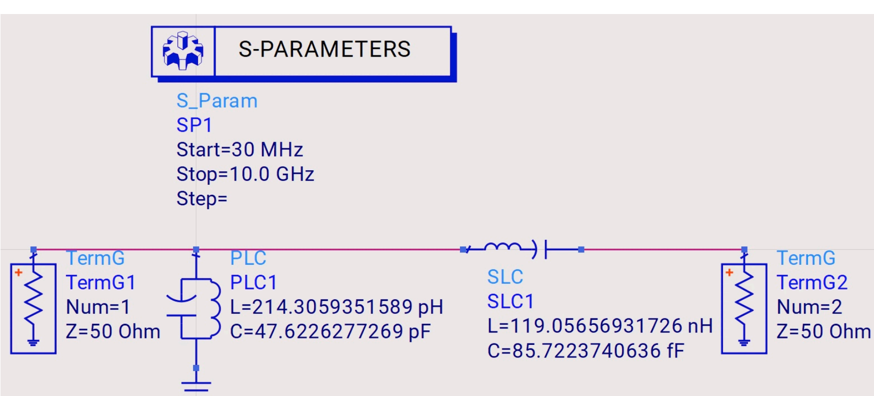
such that $L_1 = \dfrac{wR_0}{\omega_0 C} = \dfrac{0.06(50)}{1.4142\,\omega_0} = 214.3059351589\ \text{pH}$

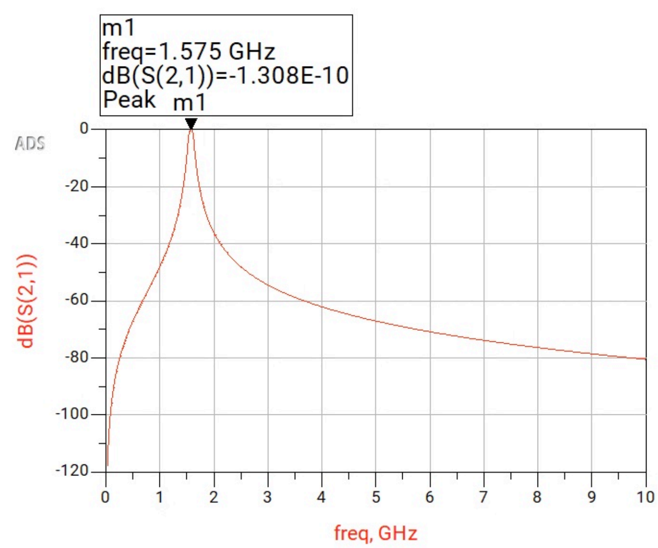
and $C_1 = \dfrac{C}{\omega_0 w R_0} = \dfrac{1.4142}{\omega_0\, 0.06(50)} = 47.6226277269\ \text{pF}$

$L = 1.4141$ will get transformed to a series LC

such that $L_2 = \dfrac{LR_0}{\omega_0 w} = \dfrac{50(1.4142)}{0.06\,\omega_0} = 119.05656931726\ \text{nH}$

and $C_2 = \dfrac{w}{\omega_0 R_0 L} = \dfrac{0.06}{\omega_0 50(1.4142)} = 85.7223740636\ \text{fF}$





Step 2:

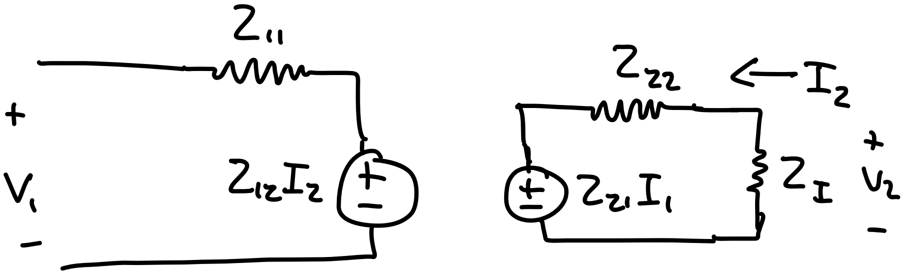
The 4 port matrix is just a more general case of the 2 port matrix, lets use the values from the 4 port matrix. Our 2 port matrix will look like so:

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \quad \text{where } Z_{11} = Z_{22} = -\frac{j}{2}(Z_{0E} + Z_{00}) \cot\theta$$

$$Z_{12} = Z_{21} = -\frac{j}{2}(Z_{0E} - Z_{00}) \csc\theta$$

(port 2 of this corresponds to port 3 of the 4 port)

Step 3:



it's actually $-\frac{V_2}{Z_I}$

$$V_1 = I_1 Z_{11} + I_2 Z_{12} \Rightarrow \frac{V_1}{I_1} = Z_{11} + \frac{I_2 Z_{12}}{I_1} \qquad I_2 = \frac{V_2}{Z_I}$$

$$\Rightarrow \frac{V_1}{I_1} = Z_{11} + \frac{V_2 Z_{12}}{Z_I I_1} = Z_{11} + \frac{Z_{12}^2}{Z_I}$$

so $Z_{in} = Z_{11} + \frac{Z_{12}^2}{Z_I}$

and $Z_{in} = Z_I$ so $\quad Z_I = Z_{11} + \frac{Z_{12}^2}{Z_I}$

$$\text{so } Z_I = -\frac{j}{2}(Z_{0E} + Z_{00}) \cot\theta + \frac{1}{Z_I}\left(-\frac{j}{2}(Z_{0E} - Z_{00}) \csc\theta\right)^2$$

$$= -\frac{j}{2}(Z_{0E} + Z_{00}) \cot 90° + \frac{1}{Z_I}\left(\frac{1}{4}(Z_{0E} - Z_{00})^2 \csc^2\theta\right)$$

$$Z_I = \frac{1}{Z_I} \cdot \frac{1}{4}(Z_{oE} - Z_{oo})^2 \csc^2\theta$$

$$Z_I^2 = -\frac{1}{4}(Z_{oE} - Z_{oo})^2 \csc^2\theta$$

$$Z_I = \frac{1}{2}(Z_{oE} - Z_{oo}) \csc\theta$$

$$Z_I = \frac{1}{2}(Z_{oE} - Z_{oo})$$

Step 4:

ABCD parameters can be chained together with matrix multiplication.

$$\begin{bmatrix} \cos\theta & iZ_o\sin\theta \\ iY_o\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -\frac{i}{J} \\ -iJ & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & iZ_o\sin\theta \\ iY_o\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & iZ_o\sin\theta \\ iY_o\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{\sin\theta}{Z_oJ} & -\frac{i\cos\theta}{J} \\ -iJ\cos\theta & JZ_o\sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sin\theta\cos\theta}{Z_oJ} + JZ_o\sin\theta\cos\theta & -\frac{i\cos^2\theta}{J} + iJZ_o^2\sin^2\theta \\ \frac{i\sin^2\theta}{JZ_o^2} - iJ\cos^2\theta & \frac{\sin\theta\cos\theta}{JZ_o} + JZ_o\sin\theta\cos\theta \end{bmatrix}$$

$$Z_I = \sqrt{\frac{B}{C}} = \sqrt{\frac{-\frac{i\cos^2\theta}{J} + iJZ_o^2\sin^2\theta}{\frac{i\sin^2\theta}{JZ_o^2} - iJ\cos^2\theta}} = \sqrt{\frac{-\frac{\cos^2\theta}{J} + JZ_o^2\sin^2\theta}{\frac{\sin^2\theta}{JZ_o^2} - J\cos^2\theta}}$$

$$= \sqrt{\frac{0 + JZ_o^2}{\frac{1}{JZ_o^2} - 0}} = \sqrt{J^2 Z_o^4} = JZ_o^2 = Z_I$$

and so $Z_I = \frac{1}{2}(Z_{oo} - Z_{oE}) = JZ_o^2$

Step 5:

For the coupled tline, if the output is open circuited, then $Z_{in} = Z_{11}$. $Z_{in} = \frac{I_1 Z_{11} + I_2 Z_{12}}{I_1}$ but open circuit output so $I_2 = 0$ so $Z_{in} = Z_{11} = -\frac{i}{2}(Z_{oE} + Z_{oo})\cot\theta$

For the inverter based design,

$$Z_{in} = \frac{A + \frac{B}{Z_L}}{C + \frac{D}{Z_L}}$$ output is open circuit so $Z_L = \infty$ so $Z_{in} = \frac{A}{C}$

$$= \frac{\frac{\sin\theta\cos\theta}{Z_o J} + JZ_o \sin\theta\cos\theta}{\frac{i\sin^2\theta}{JZ_o^2} - iJ\cos^2\theta} = \frac{i\left(\frac{\sin\theta\cos\theta}{Z_o J} + JZ_o \sin\theta\cos\theta\right)}{\frac{-\sin^2\theta}{JZ_o^2} + J\cos^2\theta}$$

Setting our two Z_{in} equal,

$$-\frac{i}{2}(Z_{oE} + Z_{oo})\cot\theta = \frac{i\left(\frac{\sin\theta\cos\theta}{Z_o J} + JZ_o \sin\theta\cos\theta\right)}{\frac{-\sin^2\theta}{JZ_o^2} + J\cos^2\theta}$$

$$\Rightarrow \frac{1}{2}(Z_{oE} + Z_{oo})\cot\theta = -\frac{\sin\theta\cos\theta\left(\frac{1}{Z_o J} + JZ_o\right)}{-\frac{\sin^2\theta}{JZ_o^2} + J\cos^2\theta}$$

$$\Rightarrow \frac{1}{2}(Z_{OE}+Z_{OO}) = -\frac{\sin\theta\cos\theta\left(\frac{1}{Z_0 J}+JZ_0\right)}{-\frac{\sin^2\theta}{JZ_0^2}+J\cos^2\theta} \cdot \frac{\sin\theta}{\cos\theta} = -\frac{\sin^2\theta\left(\frac{1}{Z_0 J}+JZ_0\right)}{-\frac{\sin^2\theta}{JZ_0^2}+J\cos^2\theta}$$

$$\Rightarrow \frac{1}{2}(Z_{OE}+Z_{OO}) = \frac{-\frac{1}{Z_0 J}-JZ_0}{-\frac{1}{JZ_0^2}+0} = \frac{\frac{1}{Z_0 J}}{\frac{1}{JZ_0^2}} + \frac{JZ_0}{\frac{1}{JZ_0^2}} = \frac{JZ_0^2}{Z_0 J} + J^2 Z_0^3$$

$$\Rightarrow \frac{1}{2}(Z_{OE}+Z_{OO}) = Z_0 + J^2 Z_0^3$$

So, now we have two equations:

$$\frac{1}{2}(Z_{OE}+Z_{OO}) = Z_0(1+J^2Z_0^2)$$

$$\frac{1}{2}(Z_{OE}-Z_{OO}) = JZ_0^2$$

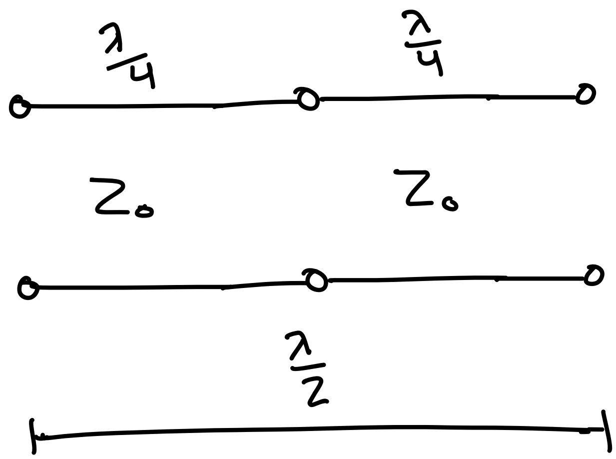
$$\begin{array}{rrcl}
\frac{1}{2}Z_{OE} & +\frac{1}{2}Z_{OO} & = & Z_0(1+J^2Z_0^2) \\
\frac{1}{2}Z_{OE} & -\frac{1}{2}Z_{OO} & = & JZ_0^2 \\
\hline
Z_{OE} & +0 & = & JZ_0^2 + Z_0 + Z_0J^2Z_0^2 = Z_0(1+J+(JZ_0)^2)
\end{array}$$

so $Z_{OE} = Z_0(1+J+(JZ_0)^2)$

$$\begin{array}{rrcl}
\frac{1}{2}Z_{OE} & +\frac{1}{2}Z_{OO} & = & Z_0(1+J^2Z_0^2) \\
-\frac{1}{2}Z_{OE} & +\frac{1}{2}Z_{OO} & = & -JZ_0^2 \\
\hline
 & Z_{OO} & = & -JZ_0^2 + Z_0 + Z_0J^2Z_0^2 = Z_0(1-JZ_0+(JZ_0)^2)
\end{array}$$

so $Z_{OO} = Z_0(1-JZ_0+(JZ_0)^2)$

Step 6:



$$Z_{in} = Z_o \frac{Z_L + jZ_o \tan 2\theta}{Z_o + jZ_L \tan 2\theta}$$

$$= Z_o \frac{\frac{Z_L}{Z_L} + j\frac{Z_o}{Z_L}\tan 2\theta}{\frac{Z_o}{Z_L} + j\frac{Z_L}{Z_L}\tan 2\theta}$$

$$= Z_o \frac{1}{j\tan 2\theta}$$

$$Y_{in} = \frac{1}{Z_{in}} = \frac{j\tan 2\theta}{Z_o} = jY_o \tan 2\theta$$

$$= jY_o \tan\pi = 0 =$$

$$Y_{in} = 0$$

$$B = Y_o \tan\left(\frac{\omega}{v} l\right)$$

$$\frac{dB(\omega)}{d\omega} = Y_o \sec^2\left(\frac{\omega}{v} l\right)\frac{l}{v}$$

$$@\ \omega = \omega_o \quad \beta l = \frac{\omega_o}{v} l = \frac{\omega_o}{v}\frac{\lambda}{2} = \pi$$

$$\left.\frac{dB(\omega)}{d\omega}\right|_{\omega=\omega_o} = Y_o \sec^2(\pi)\frac{l}{v} = Y_o \sec^2(\pi)\frac{\lambda}{2v}$$

$$= Y_o \frac{\lambda}{2v} = Y_o \frac{2\pi v}{\omega_o}\cdot\frac{1}{2v} = Y_o \frac{\pi}{\omega_o}$$

so $$\left.\frac{dB(\omega)}{d\omega}\right|_{\omega=\omega_o} = Y_o \frac{\pi}{\omega_o}$$

and $$C_{eff} = \frac{1}{2}\left.\frac{dB(\omega)}{d\omega}\right|_{\omega=\omega_o} = \boxed{Y_o \frac{\pi}{2\omega_o} = C_{eff}}$$

$$\boxed{L_p = \frac{1}{\omega_o^2 C_{eff}}}$$

Step 7:

Assuming $Y_o = G_L = G_s$

$$J_{0,1} = \sqrt{\frac{C_i \omega_o w G_s}{g_o g_1}} = \sqrt{\frac{C_{eff}\,\omega_o w Y_o}{g_o g_1}} = \sqrt{\frac{\frac{\pi}{2\omega_o}\omega_o w Y_o^2}{g_o g_1}}$$

$$= Y_o\sqrt{\frac{\pi w}{2 g_o g_1}}$$

so $$J_{0,1} = \frac{1}{Z_o}\sqrt{\frac{\pi w}{2 g_o g_1}} \Rightarrow \boxed{Z_o J_{0,1} = \sqrt{\frac{\pi w}{2 g_o g_1}}}$$

$$J_{i,i+1} = \omega_0 w \sqrt{\frac{C_i' C_{i+1}'}{g_i g_{i+1}}}$$ but using same C_{eff} for i & $i+1$

$$J_{i,i+1} = \omega_0 w \sqrt{\frac{C_{eff}^2}{g_i g_{i+1}}} = \omega_0 w C_{eff} \sqrt{\frac{1}{g_i g_{i+1}}} = \omega_0 w Y_0 \frac{\pi}{2\omega_0} \sqrt{\frac{1}{g_i g_{i+1}}}$$

$$= \frac{w\pi}{Z_0} \sqrt{\frac{1}{g_i g_{i+1}}} \text{ so } J_{i,i+1} = \frac{w\pi}{Z_0} \sqrt{\frac{1}{g_i g_{i+1}}} \Rightarrow \boxed{Z_0 J_{i,i+1} = \pi w \sqrt{\frac{1}{g_i g_{i+1}}}}$$

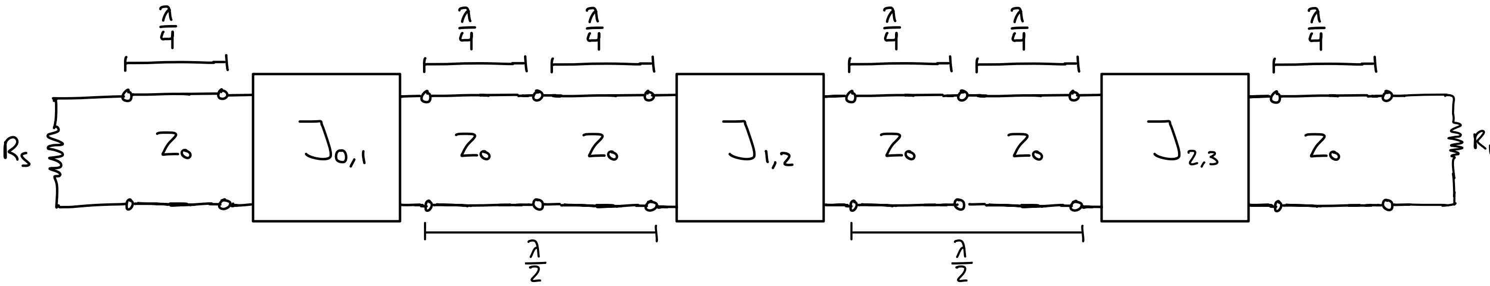
$$J_{n,n+1} = \sqrt{\frac{C_n' \omega_0 w G_L}{g_n g_{n+1}}} = \sqrt{\frac{C_{eff} \omega_0 w Y_0}{g_n g_{n+1}}} = \sqrt{\frac{Y_0^2 \frac{\pi}{2\omega_0} \omega_0 w}{g_n g_{n+1}}}$$

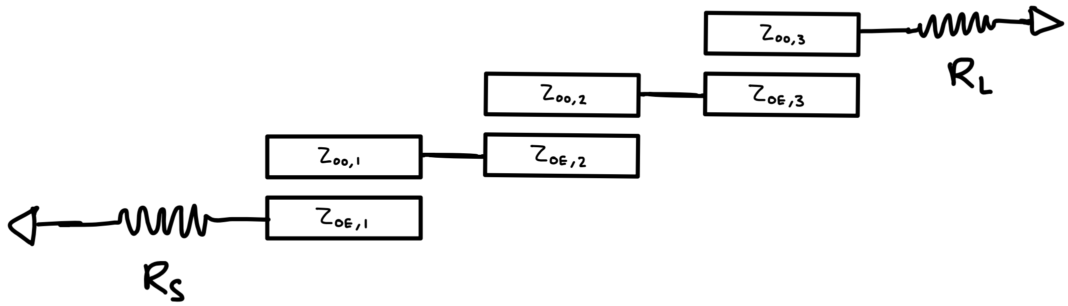
$$= Y_0 \sqrt{\frac{\pi w}{2 g_n g_{n+1}}} = \frac{1}{Z_0} \sqrt{\frac{\pi w}{2 g_n g_{n+1}}} = J_{n,n+1}$$

$$\Rightarrow \boxed{Z_0 J_{n,n+1} = \sqrt{\frac{\pi w}{2 g_n g_{n+1}}}}$$

Remember, this is under the assumption that $Z_0 = R_S = R_L$ even if it's just at the terminals as inverter based structure assumes Z_0 characteristic on left and right tlines, and the calculation with the $\frac{\lambda}{2}$ connected tline assumed a single Z_0 between the 2 connected tlines, and so chaining the inverter based structures propogates the shared Z_0 requirement.

Step 8:





$$Z_{0E,1} = Z_o\left(1 + J_{0,1}Z_o + (J_{0,1}Z_o)^2\right) = Z_o\left(1 + \sqrt{\frac{\pi w}{2g_0g_1}} + \sqrt{\frac{\pi w}{2g_0g_1}}^{\,2}\right)$$

$$= \boxed{Z_o\left(1 + \sqrt{\frac{\pi w}{2g_0g_1}} + \frac{\pi w}{2g_0g_1}\right) = Z_{0E,1}}$$

$$Z_{00,1} = Z_o\left(1 - J_{0,1}Z_o + (J_{0,1}Z_o)^2\right) = Z_o\left(1 - \sqrt{\frac{\pi w}{2g_0g_1}} + \sqrt{\frac{\pi w}{2g_0g_1}}^{\,2}\right)$$

$$= \boxed{Z_o\left(1 - \sqrt{\frac{\pi w}{2g_0g_1}} + \frac{\pi w}{2g_0g_1}\right) = Z_{00,1}}$$

$$Z_{0E,2} = Z_o\left(1 + J_{1,2}Z_o + (J_{1,2}Z_o)^2\right) = Z_o\left(1 + \pi w\sqrt{\frac{1}{g_1g_2}} + \left(\pi w\sqrt{\frac{1}{g_1g_2}}\right)^2\right)$$

$$= \boxed{Z_o\left(1 + \frac{\pi w}{\sqrt{g_1g_2}} + \frac{\pi^2 w^2}{g_1g_2}\right) = Z_{0E,2}}$$

$$Z_{00,2} = Z_o\left(1 - J_{1,2}Z_o + (J_{1,2}Z_o)^2\right) = Z_o\left(1 - \pi w\sqrt{\frac{1}{g_1g_2}} + \left(\pi w\sqrt{\frac{1}{g_1g_2}}\right)^2\right)$$

$$= \boxed{Z_o\left(1 - \frac{\pi w}{\sqrt{g_1g_2}} + \frac{\pi^2 w^2}{g_1g_2}\right) = Z_{00,2}}$$

$$Z_{0E,3} = Z_0\left(1 + J_{2,3}Z_0 + (J_{2,3}Z_0)^2\right) = Z_0\left(1 + \sqrt{\frac{\pi w}{2g_2g_3}} + \sqrt{\frac{\pi w}{2g_2g_3}}^{\,2}\right)$$

$$= \boxed{Z_0\left(1 + \sqrt{\frac{\pi w}{2g_2g_3}} + \frac{\pi w}{2g_2g_3}\right) = Z_{0E,3}}$$

$$Z_{0O,3} = Z_0\left(1 - J_{2,3}Z_0 + (J_{2,3}Z_0)^2\right) = Z_0\left(1 - \sqrt{\frac{\pi w}{2g_2g_3}} + \sqrt{\frac{\pi w}{2g_2g_3}}^{\,2}\right)$$

$$= \boxed{Z_0\left(1 - \sqrt{\frac{\pi w}{2g_2g_3}} + \frac{\pi w}{2g_2g_3}\right) = Z_{0O,3}}$$